Adiabatic Approximation from a Renormalization Group Inspired Method

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We show how a method inspired in renormalization group techniques can be useful for deriving Hamiltonians in the adiabatic approximation in a systematic way.

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Two related renormalization group schemes for Hamiltonians have been independently proposed in the last years by Wegner [1] and Glazek and Wilson [2] in the context of condensed matter and field theory, respectively. These nonperturbative methods are useful for diagonalizing or renormalizing a given Hamiltonian and are based on a continuous sequence of (infinitesimal) unitary transformations applied to the system Hamiltonian. The key point in these approaches is that the unitary transformations are chosen in such a way that off-diagonal matrix elements becomes smaller at each step of the sequence. The outset of this procedure is a set of flow equations whose solution yields (in the best case) a (almost) diagonalized Hamiltonian. These methods have been succesfully applied to various problems in condensed matter and light front field theory [3].

In this letter we take a look at the origins of this kind of renormalization transformations, i.e., the work of Fröhlich [4] (see also [5]). He gave an explanation for the effective interaction between electrons in BCS theory of superconductivity by eliminating electron-phonon interactions using a unitary transformation, which can be viewed as the "one-step" formulation of the method mentioned above. We show how Fröhlich method can be used as an approach to the usual adiabatic approximation used in quantum optics. To that end we will use as a toy model the Dicke Hamiltonian [7] which describes a collection of two-level atoms interacting with a quantum field.

In the method of Fröhlich a similarity transformation is performed on the Hamiltonian such that the transformed Hamiltonian has the (troublesome) off-diagonal elements equal to zero. Since similarity transformations preserve eigenvalues, the transformed Hamiltonian has the same spectrum as the original one. Thus the new Hamiltonian is given by

$$H_U = UHU^\dagger. \tag{1}$$

Let us write the unitary operator as

$$U = e^{-i\Omega} \tag{2}$$

where Ω is defined as a perturbative series in a coupling constant. If we separate the Hamiltonian into its free and interacting parts we have

$$H_U = H_0 + (H_I + i[H_0, \Omega]) + i[H_I, \Omega] - \frac{1}{2}[[H_0, \Omega], \Omega] + \cdots \tag{3}$$

We must look for an operator Ω such that it eliminates the effect of the Hamiltonian H_I to lowest order, that is, such that the transformed Hamiltionian is completely diagonal to that order. This means, we must impose the condition

$$H_I + i[H_0, \Omega] = 0 \tag{4}$$

which in turn implies that when this condition is satisfied, H_U can be written as

$$H_U = H_0 + \frac{i}{2}[H_I, \Omega] + \cdots \tag{5}$$

To show how the adiabatic approximation is obtained from it we use as a toy model the Dicke Hamiltonian

$$H = \omega_0 a^\dagger a + \omega_1 S_z + g\ \left(aS_+ + a^\dagger S_-\right) \tag{6}$$

where a and $a^\dagger$ are the usual field operators and the collective atomic operators satisfy

$$[S_+, S_-] = 2S_z, \qquad [S_z, S_\pm] = \pm S_\pm \tag{7}$$

In the limit of large detuning, $\omega_1 \gg \omega_0$, we can adiabatically eliminate the transitions among different eigenstates of S_z. That is, we want to decouple the low frequency modes from the high frequency modes (this can be done at least perturbatively by diagonalizing with a suitable unitary operator). By replacing H_0 and H_I into eqs. (3) and (4) we have

$$[\omega_0 a^\dagger a + \omega_1 S_z, \Omega] = ig\ \left\{aS_+ + a^\dagger S_-\right\} \tag{8}$$

From the structure of this equation and the algebra of the operators involved, one can easily infer that Ω must be necessarily of the form

$$\Omega = g\left\{\alpha a S_+ + \beta a^\dagger S_-\right\} + \Omega' \tag{9}$$

where Ω' is any operator that commutes with the free Hamiltonian H_0. That is, the solution to eq. (4) is always of the form

$$\Omega = \overline{\Omega} + \Omega' \tag{10}$$

The fact that Ω' remains undetermined is harmless. In effect, we have

$$H_U = e^{-i\Omega' - i\overline{\Omega}} H e^{i\Omega' + i\overline{\Omega}} \tag{11}$$

but we can always perform a unitary transformation on H_U such that[1]

$$H'_U = e^{-i\overline{\Omega}} e^{-i\Omega'} H e^{i\Omega'} e^{i\overline{\Omega}}. \tag{12}$$

which means that an unknown unitary transfomation is acting on the original Hamiltonian H:

$$H' = e^{-i\Omega'} H e^{i\Omega'} \tag{13}$$

Since a unitary transformation does not change the eigenvalues of the Hamiltonian, we can always make the choice of working either with H' or H. By choosing to work with H is equivalent in practice as imposing the condition $\Omega' = 0$. We do this in the following.

Therefore, after replacing eq. (9) into eq. (8) we get

$$\Omega = i\frac{g}{\Delta}\left\{aS_+ - a^\dagger S_-\right\} \tag{14}$$

where $\Delta = \omega_1 - \omega_0$.

Also, the $\mathcal{O}(g^2)$ contribution

$$H^{(2)} = \frac{i}{2}[H_I, \Omega] \tag{15}$$

is readily obtained from (6) and (14):

$$H^{(2)} = \frac{g^2}{\Delta}\left\{2S_z a^\dagger a + S_+ S_-\right\} \tag{16}$$

which implies that the correct total Hamiltonian to this order in perturbation theory is

$$H_U = \omega_0 a^\dagger a + \omega_1 S_z + 2\frac{g^2}{\Delta} a^\dagger a S_z + \frac{g^2}{\Delta}\left(S^2 - S_z^2 + S_z\right) \tag{17}$$

This Hamiltonian can also be derived by using density matrix techniques in the adiabatic approximation but the calculation is too much involved [8]. It has also been derived by using unitary transformations in [9], but the way of obtaining it was based on the calculation done in [8] and then performing a guess. In that way they obtained

$$H = \omega_0 a^\dagger a + \omega_1 S_z + 2\frac{g^2}{\Delta} a^\dagger a S_z + \frac{g^2}{\Delta}\left(S^2 - S_z^2 + S_z\right) + \frac{g^2}{2\Delta}\left(S_+^2 + S_-^2\right) \tag{18}$$

where the last term had to be discarded by making a further approximation.

One could have also used a sequence of discrete unitary transformations, as done e.g. in [6]. After a lengthy calculation one gets

$$H_U = \omega_0 a^\dagger a + \omega_1 S_z + (\frac{g^2}{\omega_1} + \frac{g^2\omega_0}{\omega_1^2} + \frac{g^2\omega_0^2}{\omega_1^3})\left\{2S_z a^\dagger a + S_+ S_-\right\} + \frac{g\ \omega_0^2}{\omega_1^2}\left\{aS_+ + a^\dagger S_-\right\} \tag{19}$$

where the last term must be removed by another unitary transformation and we recognize in the remaining $\mathcal{O}(g^2)$ term part of the geometric series corresponding to the expansion of g^2/Δ.

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[1] This can be easily proved by following the standard steps one makes when deriving the Baker-Campbell-Hausdorff or related (e.g Zassenhaus) formulas.

In this letter we have shown that it is possible to use a renormalization group inspired method to derive Hamiltonians in the adiabatic, large detuning, approximation in a straightforward and systematic way, avoiding the appearance of extra terms that should not be there. It must be stressed that although the model we used to show the method is simple, the procedure can be implemented with no difficulties in more realistic models. The crucial step will always be to find the solution to the operator equations that appear in the procedure, like eq. (4) above. This is done by inspection, which in practice it is not difficult to do.

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